Shifted small deviations and Chung LIL for symmetric alpha-stable processes.

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Abstract

Let X_α be a symmetric α-stable Lévy process with $\alpha \in (1,2)$. We consider small ball probabilities of the following type $\mathbf{P}\{\|X_\alpha - \lambda f\| < r\}$ as $r \to 0$ and $\lambda r^{\alpha-1} \to 0$ or $\lambda r^{\alpha-1} = c$, $c > 0$, where $\|\cdot\|$ is the sup-norm and f is any continuous function which starts at 0. We obtain an exact rate of decrease for these probabilities including constants.

Using these small ball estimates, we derive a functional LIL for X_α with continuous attracting functions. It occurs that the a.s. limit set of the family $\left\{\frac{X_\alpha(T\cdot)}{T^{1/\alpha}h(T)}\right\}_{T>0}$ is equal to the set of all continuous functions (which start at 0), under certain choice of scaling function $h(T)$.

Keywords: Chung LIL, Strassen LIL, small ball probabilities, Lévy processes, stable Lévy processes.

Introduction

We are interested in the probabilities that a càdlàg process $X(t), t \in [0,1]$ hits an arbitrary small ball, i.e., $\mathbf{P}\{X \in \mathrm{B}(f,r)\}$, where $\mathrm{B}(f,r)$ is a ball (in the Skorokhod metric or in the uniform metric) of radius $r > 0$, $(r \to 0)$, and of center f, which is an arbitrary element of the Skorokhod space $D[0,1]$.

If the shift function (center) f has jumps, i.e., $f \in D[0,1] \setminus C[0,1]$, then the problem is delicate. If the process X has no fixed-time jumps, what holds in most of practically important cases, the uniform small balls are obviously empty. Hence, one has to deal with the Skorokhod topology. We don't know any results on probabilities of small balls in the Skorokhod topology. This is a subject of future research.

However, if we assume that the shift f is a continuous function, then there is a sense to consider small balls in the uniform topology (as well as in the Skorokhod topology). Dealing with uniform balls and the uniform topology is more usual and there are already some results in this direction.

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In the sequel, by $\|\cdot\|$ we denote the uniform norm, and by $B(f, r)$ a ball of radius r and of center f in the uniform metric.

Aurzada and Dereich (see [AD08]) elaborate a method that allows to estimate $\mathbf{P}\{\|X\| < r\}$, where X is an arbitrary Lévy process. So, they deal with time-homogeneous processes, i.e., the shift f is the identity function multiplied by a constant. Since we are interested in applications to the functional law of the iterated logarithm (functional LIL), we need to study similar probabilities but with arbitrary shift functions. Thus, in general, we deal with time-inhomogeneous (additive) processes.

In this article, we focus on symmetric α-stable Lévy processes X_α.

Concerning the LIL for these processes, there are significant differences from the gaussian and the pre-gaussian cases. Namely, Limsup LIL doesn't exist, i.e., there is no such a scaling function $\varphi(\cdot)$ that $0 < \limsup_{t\to\infty} |X_\alpha(T)|/\varphi(T) < \infty$. Instead, there is an integral test for φ (see Fact 2 below) that says whether this limit is equal to 0 or to ∞.

In spite of that, there is a Liminf LIL statement by Taylor [Tay67]:

$$\liminf_{T\to\infty} \frac{\|X_\alpha(T\cdot)\|}{(T/\log\log T)^{1/\alpha}} = K_\alpha^{1/\alpha} \quad a.s.,$$

where K_α is a positive constant (the same as in (3) below).

Based on these two facts, we are looking for a functional LIL for X_α under those scaling functions φ, which are bigger than $(T/\log\log T)^{1/\alpha}$. For example, if $\varphi(T)\cdot(T/\log\log T)^{-1/\alpha} \to \infty$, then

$$\liminf_{T\to\infty} \left\| \frac{X_\alpha(T\cdot)}{\varphi(T)} \right\| = 0 \quad a.s.,$$

what means that the family of scalings $\left\{\frac{X_\alpha(T\cdot)}{\varphi(T)}\right\}_{T>0}$ has at least one a.s. limit point under uniform convergence, this is the zero function. If, moreover, the integral test gives 0, then this is the only a.s. limit point.

In this article, we study the a.s. limit sets of the family under these scaling functions φ that ensure ∞ in the integral test. In Theorems 4 and 5, we obtain that if $\varphi(T) \in ((T/\log\log T)^{1/\alpha}, T^{1/\alpha}\log\log T^{1-1/\alpha})$, then the a.s. limit set of $\left\{\frac{X_\alpha(T\cdot)}{\varphi(T)}\right\}_{T>0}$ in the uniform topology is equal to the set of all continuous functions which start at 0.

The border line $\varphi(T) = C\cdot(T/\log\log T)^{1/\alpha}, C > 0$ is studied in Theorem 3, which shows that the scaling is too small and the trajectories stop a.s. clustering near continuous functions, i.e., the a.s. limit set is empty.

Of course, it is interesting to understand what happens when the scaling function is close to the border of the integral test. It requires additional study.

The article is structured as follows.

In section 1, we obtain small deviation estimates for $\mathbf{P}\{\|X_\alpha - \lambda f\| < r\}$ as $r \to 0$, first under $\lambda r^{\alpha-1} \to 0$, see Theorem 1, then under $\lambda r^{\alpha-1} = c$, $c > 0$, see Theorem 2. For the proof we use the Girsanov theorem for additive processes, that is provided in section 0.1.

We start section 2 with a detailed review on the LIL for stable Lévy processes and discuss a Baldi-Royonette result (see [BR92]) for the Wiener process that describes a parallel situation with the main results of this article. In Theorems 3, 4 and 5, we get the a.s. limit set (subset) of the families $\left\{\frac{X_\alpha(T\cdot)}{T^{1/\alpha}h(T)}\right\}_{T>0}$, when $h(T) \in [(\log\log T)^{-1/\alpha}, (\log\log T)^{1-1/\alpha}]$. We obtain these results together with the rates of convergence to the limit functions.

0.1 Notations and tools.

Let $\mathcal{C} = \{f \in C[0,1] : f(0) = 0\}$.
By $AC[0,1]$ we denote the set of all absolutely continuous functions on $[0,1]$.
Put $\mathbf{H} = \{f \in AC[0,1] :\ f' \in L_2,\ f(0) = 0\}$.
We use notations from the Sato monograph [Sat99], to introduce a Lévy process. In this article, we deal with processes of finite expectation, therefore it is convenient to define a Lévy process X by its centered triplet $(\sigma^2, \Lambda(dx), \gamma)_1$, where σ^2 is the variance of the gaussian component (here, $\sigma^2 = 0$), $\Lambda(dx)$ is the Lévy measure and γ is the expectation of $X(1)$. Here is the corresponding Lévy-Ito decomposition

$$X(t) = \gamma t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\bar{\mathcal{N}}_\Lambda(dx, ds),$$

were $\mathcal{N}_\Lambda(dx, dt)$ is a Poisson measure corresponding to the Lévy measure Λ, and $\bar{\mathcal{N}}_\Lambda(dx, dt) = \mathcal{N}_\Lambda(dx, dt) - \Lambda(dx)dt$ is the centered Poisson measure.

If $\gamma = 0$, we call the corresponding Lévy process a $(\Lambda, 0)$-Lévy martingale.

By additive processes, we mean time-inhomogeneous processes with independent increments, that start at 0. The distributions of the processes with finite expectations are specified by the centered triplets $(0, \Lambda(dx, dt), \gamma(t))_1$, $\gamma \in L_1$. The corresponding Lévy-Ito decomposition is

$$X(t) = \int_0^t \gamma(s)ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\bar{\mathcal{N}}_\Lambda(dx, ds).$$

Denote by P_ξ the distribution of the process ξ in $D[0,1]$.

In the next section, we will need the following particular case of the Girsanov theorem, see Theorem 3.24 from [JS03], see also [LS02], Theorem 2:

Fact 1 (*The Girsanov transform for additive processes with finite expectations*)

Let ξ be an additive process defined by the centered triplets $(0, \Lambda(dx, dt), \gamma(t))_1$, $\gamma \in L_1[0,1]$. Suppose there exists $\theta(\cdot,\cdot) : \mathbb{R}\setminus\{0\} \times [0,1] \to \mathbb{R}$ such that

$$\int_0^1 \int_{\mathbb{R}\setminus\{0\}} \left(e^{\theta(x,s)/2} - 1\right)^2 \Lambda(dx, ds) < \infty. \tag{1}$$

Then the distribution of an additive process ξ_θ defined by

$$\left(0,\ e^{\theta(x,s)}\Lambda(dx,dt),\ \gamma(t)+\left(\int_0^t\int_{\mathbb{R}}(e^{\theta(x,s)}-1)x\Lambda(dx,ds)\right)'_t\right)_1$$

is equivalent to the distribution of ξ, i.e., $P_\xi \sim P_{\xi_\theta}$ and the density transformation formula is of the form:

$$\frac{dP_{\xi_\theta}}{dP_\xi}(\xi(\cdot)) = \exp\left\{-\int_0^1\int_{\mathbb{R}\setminus\{0\}}\left(e^{\theta(x,s)}-1-\theta(x,s)\right)\Lambda(dx,dt)+\right.$$
$$\left.+\int_0^1\int_{\mathbb{R}\setminus\{0\}}\theta(x,s)\bar{\mathcal{N}}_\Lambda(dx,dt)(\cdot)\right\},\quad P_\xi\text{ -a.e.}$$

Comments:

1. Condition (1) guarantees existence of the integrals and the properties of Lévy measure for $e^{\theta(x,s)}\Lambda(dx,dt)$.
2. Note that if there exists θ^* such that

$$\int_0^t\int_{\mathbb{R}\setminus\{0\}}(e^{\theta^*(x,s)}-1)x\Lambda(dx,ds) = -\int_0^t\gamma(s)\,ds,\quad \forall t\in[0,1], \tag{2}$$

then the transformed process is a martingale.

1 Shifted small ball probabilities for symmetric α-stable processes

1.1 "Small" shifts.

Let X_α be a symmetric α-stable Lévy process, $\alpha\in(1,2)$. The aim of this section is to estimate shifted small ball probabilities for these processes, unlike the centered small ball probabilities that were studied in [Mog74]

$$\mathbf{P}\{X_\alpha\in \mathrm{B}(0,r)\} = \exp\{-K_\alpha r^{-\alpha}(1+o(1))\}, \tag{3}$$

where $0<K_\alpha<\infty$, it depends just on the process X_α. This constant is equal to the first eigenvalue of the fractional Laplacian (cf. [ZRK07]), the explicit expression for K_α is still not found.

Theorem 1 *For all $f\in\mathcal{C}$ and $\lambda>0$, $r>0$ such that $\lambda r^{\alpha-1}\to 0$, $r\to 0$ we have*

$$\mathbf{P}\ \{\|X_\alpha(\cdot)-\lambda f(\cdot)\|<r\} = \exp\left\{-K_\alpha\, r^{-\alpha}(1+o(1))\right\}.$$

Comments:

1. In this theorem, we consider relatively small λ, namely $\lambda = o(r^{-(\alpha-1)})$. The case when λ is finite is included in this part of the result.

2. Notice that the estimate is similar with the estimate (3) for centered small deviations. The leading term of the asymptotic estimate is not sensitive for f, the dependence on f is hidden in the rest term.

Proof.

Upper bound: Using the Anderson inequality which holds for symmetric processes (cf. [LRZ95], [BK86]), and taking into account (3), we obtain

$$\mathbf{P}\left\{X_\alpha \in \mathrm{B}(\lambda \cdot f, r)\right\} \le \mathbf{P}\left\{X_\alpha \in \mathrm{B}(0, r)\right\} \le \exp\{-K_\alpha r^{-\alpha}(1+o(1))\}.$$

Lower bound: We modify an approach from [Shm06].

Take $f \in \mathbf{H}$. By using self-similarity, we can write

$$\mathbf{P}\left\{\|X_\alpha - \lambda\, f\| < r)\right\} = \mathbf{P}\left\{\|\xi_1\| < r\rho^{1/\alpha}\right\},$$

where ξ_1 is a Lévy process with the centered triplet $(0, \rho\,|x|^{-1-\alpha}dx\,dt,\ -\lambda\rho^{1/\alpha}f'(t))_1$ and ρ is an arbitrary positive real number that we are free to choose.

Using Fact 1, we have that an additive process ξ_2 with the generating triplet

$$\left(0,\ \rho e^{\theta(x,t)}\frac{dx}{|x|^{1+\alpha}}dt, 0\right)_1,$$

where $\theta(x,t) = \log\left(1 + \lambda\rho^{-\frac{\alpha-1}{\alpha}} \cdot \frac{2-\alpha}{2}\, f'(t)\, x\, 1_{\{|x|<1\}}\right)$ has distribution P_{ξ_2} equivalent to P_{ξ_1} and it is a martingale. Note that it is defined correctly if

$$\lambda\rho^{-\frac{\alpha-1}{\alpha}} \cdot \frac{2-\alpha}{2}|f'(t)| < 1 \quad \textit{for almost all} \quad t \in [0,1].$$

If we assume

$$\rho:\ \lambda\rho^{-(\alpha-1)/\alpha} \to 0, \tag{4}$$

then this condition holds for large enough ρ. This will be the first restriction we impose on ρ. We continue

$$\begin{aligned}
&\mathbf{P}\left\{X_\alpha - \lambda\, f \in B(0,r)\right\} = \int_{B(0, r\rho^{1/\alpha})} \frac{dP_{\xi_1}}{dP_{\xi_2}} dP_{\xi_2} = \\
&= \exp\left\{-\rho\int_0^1\int_{|\ell|<1} \Psi\left(\lambda\rho^{-\frac{\alpha-1}{\alpha}} \cdot \frac{2-\alpha}{2}\, f'(t)\, x\right)\frac{dx}{|x|^{1+\alpha}}dt\right\} \times \\
&\mathbf{E}\exp\left\{-\int_0^1\int_{|x|<1}\log\left(1 + \lambda\rho^{-\frac{\alpha-1}{\alpha}} \cdot \frac{2-\alpha}{2}\, f'(t)\, x\right)\bar{\mathcal{N}}_{\xi_2}(dx, dt)\right\} 1_{\{\|\xi_2\| < r\rho^{1/\alpha}\}} = \\
&= D_\rho \times S_\rho,
\end{aligned}$$

where $\Psi(u) = (1+u)\log(1+u) - u = \frac{u^2}{2}(1+o(1))$ as $u \to 0$.

Deterministic term simplification.

$$D_\rho = \exp\left\{-\frac{2-\alpha}{4}\lambda^2\rho^{(2-\alpha)/\alpha}\int_0^1 f'(t)^2 dt\ (1+o(1))\right\}.$$

Stochastic term simplification. By the Jensen inequality, we get rid of the stochastic term of the density transformation formula

$$S_\rho \geq \exp\left\{-\mathbf{E}_{\mathbf{P}'}\int_0^1\int_{|x|<1}\log\left(1+\lambda\rho^{-\frac{\alpha-1}{\alpha}}\cdot\frac{2-\alpha}{2}f'(t)\,x\right)\bar{\mathcal{N}}_{\xi_2}(dx,dt)\right\}\mathbf{P}\left\{\xi_2\in B(0,r\rho^{1/\alpha})\right\} =$$
$$= \mathbf{P}\left\{\|\xi_2\| < r\rho^{1/\alpha}\right\},$$

where $\mathbf{P}' : \frac{d\mathbf{P}'}{d\mathbf{P}} = 1_{\{\|\xi_2\|<r\rho^{1/\alpha}\}}(\mathbf{P}\left\{\|\xi_2\| < r\rho^{1/\alpha}\right\})^{-1}$. It is left just to treat the small ball probability $\mathbf{P}\left\{\|\xi_2\| < r\rho^{1/\alpha}\right\}$ for the time-inhomogeneous martingale ξ_2.

Homogenization. It is clear that $\xi_2(\cdot) \stackrel{d}{=} \xi(\rho\cdot)$, where ξ is a Lévy process with the centered triplet

$$\left(0,\ \left(1+\lambda\rho^{-\frac{\alpha-1}{\alpha}}\cdot\frac{2-\alpha}{2}f'(t)\,x\,1_{\{|x|<1\}}\right)\frac{dx}{|x|^{1+\alpha}}dt, 0\right)_1.$$

We can represent the process as a sum of independent processes $\xi(\cdot) \stackrel{d}{=} \zeta_1(\cdot)+\zeta_2(\cdot)$, where ζ_1 is a Lévy process generated by the centered triplet

$$\left(0, \left(1-|x|1_{\{|x|<1\}}\right)\frac{dx}{|x|^{1+\alpha}}, 0\right)_1,$$

ζ_2 is an additive process generated by

$$\left(0,\ \left(|x|+\lambda\rho^{-\frac{\alpha-1}{\alpha}}\cdot\frac{2-\alpha}{2}f'(t)\,x\right)1_{\{|x|<1\}}\frac{dx}{|x|^{1+\alpha}}dt, 0\right)_1.$$

Taking into account this decomposition, for any $\delta\in(0,1)$ we can write

$$\mathbf{P}\left\{\|\xi(\rho\cdot)\| < r\rho^{1/\alpha}\right\} \geq \quad \mathbf{P}\left\{\|\zeta_1(\rho\cdot)\| < (1-\delta)r\rho^{1/\alpha}\right\}\mathbf{P}\left\{\|\zeta_2(\rho\cdot)\| < \delta\cdot r\rho^{1/\alpha}\right\}.$$

Let us treat each of the probabilities separately.

Using results of section 8.2.4 of [BGT89], one can prove that the process ζ_1 belongs to the domain of normal attraction of X_α, i.e.,

$$\frac{\zeta_1(\rho\cdot)}{\rho^{1/\alpha}} \stackrel{d}{\Rightarrow} X_\alpha(\cdot) \quad as \quad \rho\to\infty,$$

where "$\stackrel{d}{\Rightarrow}$" means convergence in distribution.

By [Mog74] and a slight generalization of his result by [Rus07], we know the following:

Proposition 1 *For any Lévy process X which is a martingale and belongs the normal domain of attraction of a strictly α-stable Lévy process X_α, we have*

$$\mathbf{P}\left\{\frac{\|X(\rho\cdot)\|}{\rho^{1/\alpha}}<r\right\}=\exp\left\{-\frac{K_\alpha}{r^\alpha}(1+o(1))\right\},$$

that holds as $r\to 0$ and $r\rho^{1/\alpha}\to\infty$. The constant K_α is as in (3).

In particular, if X is from the normal domain of attraction to the Wiener process, then $K_2=\pi^2/8$.

Thus, assuming

$$\rho:\quad r\rho^{1/\alpha}\to\infty, \tag{5}$$

we have

$$\mathbf{P}\left\{\left\|\frac{\zeta_1(\rho\cdot)}{\rho^{1/\alpha}}\right\|<(1-\delta)r\right\}=\exp\left\{-\frac{K_\alpha}{r^\alpha(1-\delta)^\alpha}(1+o(1))\right\}.$$

In its turn, ζ_2 could be decomposed into the sum of processes with only positive and only negative jumps, $\zeta_2\stackrel{d}{=}\zeta^++\zeta^-$, where $\zeta^\pm$ are generated by

$$\left(0,\left(1\pm\lambda\rho^{-\frac{\alpha-1}{\alpha}}\cdot\frac{2-\alpha}{2}\,f'(t)\right)1_{\{\pm x\in(0,1)\}}\frac{dx}{(\pm x)^\alpha}dt,0\right)_1.$$

This decomposition give us

$$\mathbf{P}\left\{\|\zeta_2(\rho\cdot)\|<\delta\cdot r\rho^{1/\alpha}\right\}\geq\mathbf{P}\left\{\|\zeta^+(\rho\cdot)\|<(\delta/2)r\rho^{1/\alpha}\right\}\mathbf{P}\left\{\|\zeta^-(\rho\cdot)\|<(\delta/2)r\rho^{1/\alpha}\right\}.$$

Now, we need the following lemma, that allows to switch to homogeneous processes.

Lemma 1 *Let Λ be a Lévy measure such that $\int_{|x|>1}x\Lambda(dx)<\infty$ and $\mu(\cdot)\in AC[0,1]$.*

If η is an additive process specified by the generating triplet $(0,\mu'(t)dt\,\Lambda(dx),0)_1$ and ζ is a Lévy process specified by the generating triplet $(0,(\int_0^1\mu(t)dt)\,\Lambda(dx),0)_1$, then

$$\|\eta\|\stackrel{d}{=}\|\zeta\|.$$

Proof. By Lévy-Khintchine formula we have

$$\|\eta\|\stackrel{d}{=}\sup_{t\in[0,1]}\left|X\left(\int_0^t\mu(s)ds\right)\right|=\sup_{t\in[0,\int_0^1\mu(s)ds]}\left|X\left(\frac{\int_0^t\mu(s)ds}{\int_0^1\mu(s)ds}\right)\right|,$$

where X is a Lévy process generated by $(0,\Lambda(dx),0)_1$.

Put $\varphi(t)=\int_0^t\mu(s)ds/\int_0^1\mu(s)ds$. Notice that $0\leq\varphi(t)\leq 1$ for all $t\in[0,1]$. Taking this into account we continue

$$\|\eta\|\stackrel{d}{=}\sup_{t\in[0,\int_0^1\mu(s)ds]}|X(\varphi(t))|\stackrel{d}{=}\sup_{t\in[0,1]}|\zeta(\varphi(t))|=\sup_{s\in[0,1]}|\zeta(s)|.$$

■

Using this lemma, we continue

$$\mathbf{P}\left\{\|\zeta_2(\rho\cdot)\| < \delta\cdot r\rho^{1/\alpha}\right\} \geq \mathbf{P}\left\{\|\eta^+(\rho\cdot)\| < (\delta/2)r\rho^{1/\alpha}\right\}\mathbf{P}\left\{\|\eta^-(\rho\cdot)\| < (\delta/2)r\rho^{1/\alpha}\right\},$$

where $\eta^\pm$ are centered subordinators generated by

$$\left(0, \left(1 \pm \lambda\rho^{-\frac{\alpha-1}{\alpha}}\cdot\frac{2-\alpha}{2}\,f(1)\right)1_{\{\pm x\in(0,1)\}}\frac{dx}{(\pm x)^\alpha}dt, 0\right)_1.$$

The processes $\eta^\pm$ have just bounded jumps, therefore they both belong to the normal domain of attraction of the Wiener process. By Proposition 1, under (5) we obtain

$$\mathbf{P}\left\{\|\eta^\pm(\rho\cdot)\| < (\delta/2)r\rho^{1/\alpha}\right\} \geq \exp\left\{-\frac{\pi^2}{2\delta^2(3-\alpha)}\,r^{-2}\rho^{-\frac{2-\alpha}{\alpha}}\left(1 \pm \lambda\rho^{-\frac{\alpha-1}{\alpha}}\cdot\frac{2-\alpha}{2}\,f(1)\right)(1+o(1))\right\}.$$

Then

$$\mathbf{P}\left\{\|\zeta_2(\rho\cdot)\| < \delta\cdot r\rho^{1/\alpha}\right\} \geq \exp\left\{-\frac{\pi^2}{\delta^2(3-\alpha)}\cdot\frac{1}{r^2\rho^{(2-\alpha)/\alpha}}(1+o(1))\right\}.$$

Thus, we obtain that for any $\delta \in (0,1)$

$$\mathbf{P}\left\{\|\xi_2\| < r\rho^{1/\alpha}\right\} \geq \exp\left\{-\frac{K_\alpha}{r^\alpha(1-\delta)^\alpha}(1+o(1)) - \frac{\pi^2}{\delta^2(3-\alpha)}\cdot\frac{1}{r^2\rho^{(2-\alpha)/\alpha}}(1+o(1))\right\}.$$

Under (5) we have

$$\mathbf{P}\left\{\|\xi_2\| < r\rho^{1/\alpha}\right\} \geq \exp\left\{-\frac{K_\alpha}{r^\alpha}(1+o(1))\right\}.$$

Collecting all the preliminary results, under (4) and (5) we get

$$\mathbf{P}\ \left\{\|X_\alpha(\cdot) - \lambda\, f(\cdot)\| < r\right\} \geq \exp\left\{-\frac{2-\alpha}{4}\lambda^2\rho^{(2-\alpha)/\alpha}\int_0^1 f'(t)^2dt\ (1+o(1)) - \frac{K_\alpha}{r^\alpha}(1+o(1))\right\}.$$

Using the condition $\lambda r^{\alpha-1} \to 0$ of the theorem (for the first time in the proof), we can find ρ obeying (4) and (5), such that $\lambda^2\rho^{(2-\alpha)/\alpha} = o(r^{-\alpha})$. For example, take $\rho^* = r^{-\alpha}(\lambda r^{\alpha-1})^{-1}$. Thus, for all $f \in \mathbf{H}$ we have

$$\mathbf{P}\ \left\{\|X_\alpha(\cdot) - \lambda\, f(\cdot)\| < r\right\} \geq \exp\left\{-\frac{K_\alpha}{r^\alpha}(1+o(1))\right\}.$$

The set $\mathbf{H}$ is dense in $\mathcal{C}$, so the result could be generalized for arbitrary $f \in \mathcal{C}$.

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Remark:

We can also exploit the same proof under $\lambda r^{\alpha-1} \to \infty$ or $\lambda r^{\alpha-1} = c,\ c > 0$ conditions.

For example, under $\lambda r^{\alpha-1} \to \infty$ for $f' \in L_\infty$ taking $\rho := (\lambda \|f'\|(2-\alpha)2(1-\epsilon))^{\alpha/(\alpha-1)}$, $\epsilon \in (0,1)$, we get

$$\mathbf{P}\left\{\|X_\alpha(\cdot) - \lambda\, f(\cdot)\| < r\right\} \geq \exp\left\{-C_1\, \lambda^{\alpha/(\alpha-1)}(1+o(1))\right\},$$

where

$$C_1 = C_1(f,\alpha) = \|f'\|^{\alpha/(\alpha-1)} \left(\frac{2-\alpha}{2}\right)^{\alpha/(\alpha-1)} \sum_{k=1}^{\infty} \frac{\int_0^1 (f'(t)/\|f'\|)^{2k} dt}{k(2k-1)(2k-\alpha)}.$$

We see that the order differs from the order of the upper estimate, and moreover, we can prove that it is not optimal. Following [AD08] we can obtain: there are constants $0 < C_1 \leq C_2 < \infty$ s.t.

$$\exp\left\{-C_2 \cdot \frac{\lambda}{r} \log \lambda r^{\alpha-1}\right\} \leq \mathbf{P}\left\{\|X_\alpha(\cdot) - \lambda\, Id(\cdot)\| < r\right\} \leq \exp\left\{-C_1 \cdot \frac{\lambda}{r} \log \lambda r^{\alpha-1}\right\},$$

where Id is the identity function on $[0,1]$.

Under $\lambda r^{\alpha-1} = c,\ c > 0$ condition, we are faced with a known open problem for processes from the domain of attraction of X_α

$$\mathbf{P}\left\{\|X(\rho\cdot)\| < c\right\} = \exp\left\{-A_\alpha(c)\rho(1+o(1))\right\},$$

$A_\alpha(c)$ is not known here.

Nevertheless, in the next section we obtain a result in this case, slightly modifying the proof.

1.2 "Middle" shifts.

Theorem 2 *For any $c > 0$, $f \in AC[0,1]\ :\ f' \in L_\infty$, $f(0) = 0$ such that*

$$\|f'(\cdot)\| < \frac{2}{2-\alpha} \cdot \frac{1}{c}, \tag{6}$$

we have

$$\exp\left\{-K_\alpha \frac{1}{r^\alpha}(1+o(1))\right\} \geq \mathbf{P}\left\{\|X_\alpha(\cdot) - c \cdot r^{-(\alpha-1)}\, f(\cdot)\| < r\right\} \geq \exp\left\{-C(\alpha)\frac{1}{r^\alpha}\right\},$$

as $r \to 0$, where

$$C(\alpha) = 2\left(\frac{1}{\alpha} + \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)(2k-\alpha)} + 24 \cdot 6^\alpha \left(\frac{1}{2-\alpha} + \frac{2^{\alpha-1}-1}{6(3-\alpha)}\right)\right),$$

and K_α is as in (3).

Proof. **Upper bound:** The Anderson inequality.

Lower bound: In this proof, we are close to [AD08] method. We start with a truncation of large jumps

$$\mathbf{P}\left\{\|X_\alpha(\cdot) - c \cdot r^{-(\alpha-1)} f(\cdot)\| < r\right\} = \mathbf{P}\left\{\|X_\alpha(\cdot) - c \cdot r^{-(\alpha-1)} f(\cdot)\| < r \mid A\right\}\mathbf{P}\left\{A\right\},$$

where A is the event that the process X_α has no jumps bigger than r, i.e., $A = \{\omega \in \Omega : \forall t \in [0,1]\ \Delta X_\alpha(t,\omega) \le r\}$. It is well-known that

$$\mathbf{P}\left\{A\right\} = \exp\left\{-\int_{|x|>r} \frac{dx}{|x|^{\alpha+1}}\right\} = \exp\left\{-\frac{2}{\alpha}\cdot r^{-\alpha}\right\}.$$

Denote by ξ_1 an additive process with the generating triplet

$$(0,\ 1_{\{|x|<r\}}|x|^{-(1+\alpha)}dx\,dt,\ -c\,r^{-(\alpha-1)}f'(t))_1.$$

Hence, we continue

$$\mathbf{P}\left\{\|X_\alpha(\cdot) - c \cdot r^{-(\alpha-1)} f(\cdot)\| < r\right\} = \exp\left\{-\frac{2}{\alpha}\cdot r^{-\alpha}\right\}\mathbf{P}\left\{\|\xi_1\| < r\right\}.$$

Using Fact 1, we obtain that ξ_2 with the generating triplet

$$\left(0,\ \left(1 + c\cdot\frac{2-\alpha}{2}f'(t)\frac{x}{r}\right)1_{\{|x|<r\}}\frac{dx}{|x|^{1+\alpha}}dt,\ 0\right)_1$$

has distribution equivalent to P_{ξ_1}. Take $\theta^*(s,t) = \log(1 + c\cdot\frac{2-\alpha}{2}f'(t)\frac{x}{r})$ in Fact 1. Note that ξ_2 is correctly defined under condition (6), and ξ_2 is a martingale.

Thus, we continue

$$\begin{aligned}
&\mathbf{P}\left\{\|\xi_1\| < r\right\} = \int_{B(0,r)} \frac{dP_{\xi_1}}{dP_{\xi_2}}(\eta)dP_{\xi_2}(\eta) = \\
&= \exp\left\{-\int_0^1\int_{|x|<r}\Psi\left(c\cdot\frac{2-\alpha}{2}f'(t)\frac{x}{r}\right)\frac{dx}{|x|^{1+\alpha}}dt\right\}\times \\
&\times \mathbf{E}\exp\left\{-\int_0^1\int_{|x|<r}\log\left(1 + c\cdot\frac{2-\alpha}{2}f'(t)\frac{x}{r}\right)\bar{N}_{\Lambda_2}(dx,dt)\right\}1_{\{\|\xi_2\|<r\}} = \\
&= D\times S,
\end{aligned}$$

where $\Psi(u) = (1+u)\log(1+u) - u = \sum_{k=2}^{\infty}(-1)^k\frac{u^k}{k(k-1)}$, as $|u|<1$.

Deterministic term simplification.

$$\begin{aligned}
D &= \exp\left\{-\frac{2}{r^\alpha}\sum_{k=1}^{\infty}\int_0^1\left(c\cdot\frac{(2-\alpha)f'(s)}{2}\right)^{2k}ds\cdot\frac{1}{2k(2k-1)(2k-\alpha)}\right\} \ge \\
&\ge \exp\left\{-\frac{2}{r^\alpha}\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)(2k-\alpha)}\right\}.
\end{aligned}$$

Stochastic term simplification. Take a probability measure $\mathbf{P}' : \frac{d\mathbf{P}'}{d\mathbf{P}} = \frac{1_{\{\|\xi_2\|<r\}}}{\mathbf{P}\{\|\xi_2\|<r\}}$, then by Jensen's inequality obtain

$$S \geq \exp\left\{-\mathbf{E}_{\mathbf{P}'} \int_0^1 \int_{|x|<r} \log\left(1 + c \cdot \frac{2-\alpha}{2} f'(t)\frac{x}{r}\right) \bar{N}_{\Lambda_2}(dx, dt)\right\} \mathbf{P}\{\|\xi_2\| < r\} = \mathbf{P}\{\|\xi_2\| < r\}.$$

To estimate the last probability, we use a proposition proved in [AD08] (see Lemma 4.1 there)

Proposition 2 *Let X be a $(\nu, 0)$-Lévy martingale with ν supported on $[-\varepsilon, \varepsilon]$, then*

$$\mathbf{P}\{\|X\| < 3\varepsilon\} \geq \exp\left\{-\left(12\,\frac{1}{\varepsilon^2}\int_{|x|<\varepsilon} x^2 \nu(dx) + 2\right)\right\}.$$

We can't use the proposition directly, because ξ_2 being a martingale nevertheless is time-inhomogeneous.

Homogenization. We decompose the process ξ_2 into a sum of independent processes, one of which ζ_1 is a Lévy process with the Lévy measure

$$\left(1 - \frac{|x|}{r}\right) 1_{\{|x|<r\}} \frac{dx}{|x|^{1+\alpha}},$$

and the second ζ_2 is an additive process with the Lévy measure

$$\left(|x| + c \cdot \frac{2-\alpha}{2} f'(t) x\right) \frac{1}{r} 1_{\{|x|<r\}} \frac{dx}{|x|^{1+\alpha}} dt,$$

we can choose shifts in such a way that both of the processes are again martingales

$$\xi_2(\cdot) \stackrel{d}{=} \zeta_1(\cdot) + \zeta_2(\cdot).$$

Taking into account this decomposition, for any $\delta \in (0, 1)$ we can write

$$\mathbf{P}\{\|\xi_2\| < r\} \geq \mathbf{P}\{\|\zeta_1\| < (1-\delta)r\}\mathbf{P}\{\|\zeta_2\| < \delta r\}.$$

Let us treat each of the probabilities separately.

Using Proposition 2, we obtain for ζ_1

$$\mathbf{P}\{\|\zeta_1\| < (1-\delta)r\} \geq \exp\left\{-\frac{1}{r^\alpha} \cdot 24 \left(\frac{3}{1-\delta}\right)^\alpha \left(\frac{1}{2-\alpha} - \frac{1-\delta}{3(3-\alpha)}\right)\right\}.$$

The sample paths of ζ_2 are of bounded variation. Thus, the following decomposition into the sum of processes with only positive and only negative jumps is possible $\zeta_2 \stackrel{d}{=} \zeta^+ + \zeta^-$, where $\zeta^\pm$ are generated by

$$\left(0,\ \left(1 \pm c \cdot \frac{2-\alpha}{2} f'(t)\right) \frac{1}{r} 1_{\{\pm x \in (0,r)\}} \frac{dx}{(\pm x)^\alpha} dt,\ 0\right)_1,$$

correspondingly. This decomposition yields

$$\mathbf{P}\left\{\|\zeta_2\| < \delta r\right\} \geq \mathbf{P}\left\{\|\zeta^+\| < (\delta/2)r\right\}\mathbf{P}\left\{\|\zeta^-\| < (\delta/2)r\right\}.$$

Using Lemma 1, we continue

$$\mathbf{P}\left\{\|\zeta^{\pm}\| < (\delta/2)r\right\} = \mathbf{P}\left\{\|\eta^{\pm}\| < (\delta/2)r\right\},$$

where $\eta^{\pm}$ are centered positive (negative) subordinators generated by

$$\left(0,\ \left(1 \pm c \cdot \frac{2-\alpha}{2} f(1)\right) \frac{1}{r}\, 1_{\{\pm x \in (0,r)\}} \frac{dx}{(\pm x)^{\alpha}},\ 0\right)_1.$$

Applying Proposition 2, obtain

$$\mathbf{P}\left\{\|\zeta_2\| < \delta r\right\} \geq \exp\left\{-\frac{1}{r^{\alpha}} \cdot \frac{24}{3-\alpha} \cdot \left(\frac{6}{\delta}\right)^{\alpha-1}\right\}.$$

For simplicity, take $\delta = 1/2$ and obtain the statement of the theorem.

■

2 Law of the Iterated Logarithm for stable Lévy processes

2.1 General information.

There are several recent works that deal with non-standard Law of the Iterated Logarithm (LIL) statements for Lévy processes and random walks, in particular, in the case when the variance of random variables is infinite, see [Ein07], [BDM08], [Sav08], [CKL00].

In this section, we collect facts related to the LIL for the stable Lévy processes. Traditionally LIL statements could be of Limsup (Strassen) or Liminf (Chung) types. Proofs of the first type of results are based on large deviation inequalities, whereas the second type of results usually needs small deviation estimates.

Limsup LIL:

One of the interpretations of the LIL is the rate of convergence in the CLT theorem. Analogue of the functional CLT theorem (invariance principle) for stable processes is:

$$\frac{X(T\cdot)}{T^{1/\alpha}L(T)} \overset{d}{\Rightarrow} X_{\alpha}(\cdot),$$

where X is a process form the domain of attraction of X_α and $L(\cdot)$ is a proper slowly varying function. If X is X_α itself, this relation is nothing more than the self-similarity property

$$\frac{X_{\alpha}(T\cdot)}{T^{1/\alpha}} \overset{d}{=} X_{\alpha}(\cdot).$$

The Marzinkevich-Zygmund LLN says

$$\frac{X_\alpha(T)}{T^{1/p}} \to 0 \quad a.s., \quad if \quad p \in (1, \alpha).$$

As for the LIL, the situation is predetermined by the following dichotomy statement (cf. Thoerem VIII.5 in [Ber96]):

Fact 2

$$\limsup_{T\to\infty} \frac{|X_\alpha(T)|}{T^{1/\alpha} h(T)} = 0 \quad a.s. \qquad or \quad = \infty \quad a.s.$$

according as

$$\int^\infty \frac{dt}{th(t)^\alpha} < \infty \qquad or \quad = \infty.$$

This fact says that the stable Lévy processes doesn't exhibit LIL behavior: there is no such a function $\varphi(\cdot)$ that $0 < \limsup_{t\to\infty} |X_\alpha(T)|/\varphi(T) < \infty$.

The statemet gives the following information on the sample paths growth at infinity: according to the integral test

$$\mathbf{P}\{\omega : \exists t_0(\omega) \ s.t. \ for \ t \in (t_0(\omega), \infty) \ \ |X_\alpha(t,\omega)| < t^{1/\alpha} h(t)\} = 1 \quad or \quad = 0.$$

That also means that the set $\{t : \ |X_\alpha(t)| > t^{1/\alpha} h(t)\}$ is a.s. bounded or unbounded according to the integral test.

For example, we can say that almost all sample paths of the process $X_\alpha(t), t \in (0, \infty)$ intersect the level $\varphi(t) = t^{1/\alpha}(\log t)^{1/\alpha}$ infinitely many times, whereas the level $\psi(t) = t^{1/\alpha}(\log t)^{\epsilon + 1/\alpha}$, $\epsilon > 0$ is overpassed just finitely many times.

In what follows, we need a limsup statement for the sup-process $M(\cdot)$ that is an increasing sample paths process defined by

$$M(T) = \sup_{s\in[0,1]} |X_\alpha(Ts)|.$$

Corollary 1 *For any* $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ *s.t.* $\int^\infty dx/\varphi(x) = \infty$ *the following holds*

$$\limsup_{T\to\infty} \frac{M(T)}{T^{1/\alpha}(\log T \cdot \varphi(\log\log T))^{1/\alpha}} = \infty \quad a.s. \tag{7}$$

Liminf LIL:

Despite the fact that the standard LIL doesn't exist, the Chung-type LIL for the stable Lévy processes holds

$$\liminf_{T\to\infty} \frac{M(T)}{(T/\log\log T)^{1/\alpha}} = K_\alpha^{1/\alpha} \quad a.s., \tag{8}$$

where K_α is as in (3). The law was discovered in [Tay67]. This statement is about the rate of moving of the sup-process away from zero. More precisely, almost all sample paths of the sup-process finitely often intersect the level $(1-c)K_\alpha^{1/\alpha}(T/\log\log T)^{1/\alpha}$ and infinitely often $(1+c)K_\alpha^{1/\alpha}(T/\log\log T)^{1/\alpha}$, for any $0<c<1$, i.e.,

$$\mathbf{P}\left\{\omega: \ \{T: M(T,\omega)<(1-c)K_\alpha^{1/\alpha}(T/\log\log T)^{1/\alpha}\} \ \ bdd\right\}=1,$$
$$\mathbf{P}\left\{\omega: \ \{T: M(T,\omega)<(1+c)K_\alpha^{1/\alpha}(T/\log\log T)^{1/\alpha}\} \ \ unbdd\right\}=1.$$

Combining (7) and (8), we can say that for any $c\in(0,1)$, any φ s.t. $\int^\infty dx/\varphi(x)=\infty$ the following holds: for T large enough

$$M(T)\in\Big((1-c)K_\alpha^{1/\alpha}(T/\log\log T)^{1/\alpha}, \quad T^{1/\alpha}(\log T)^{1/\alpha}(\varphi(\log\log T))^{1/\alpha}\Big) \quad a.s.$$

In this article, we study a generalization of these results to a functional LIL. What we get is analogous to the result of Baldi and Royonette in Gaussian case [BR92].

Baldi-Royonette result for the Wiener process: By W denote the Wiener process. Consider a family of scaling of W

$$\xi_T^\gamma(\cdot)=\frac{W(T\cdot)}{\sqrt{2T\log\log T}}\cdot\gamma(T),$$

where $\gamma: \ \mathbf{R}^+\to\mathbf{R}^+$ s.t. $\gamma(0)=0$.

Definition: Let (E,τ) be a topological space. An element $x\in E$ is called an a.s. limit point of a family $\{\xi_T\}_{T>0}$ of random elements on E, if there exists $\{T_k\}_{k=1}^\infty$, $T_k\to\infty$ such that $\xi_{T_k}(\omega)\xrightarrow{\tau}x$ as $k\to\infty$, for almost all $\omega\in\Omega$.
The set of all a.s. limit points of $\{\xi_T\}_{T>0}$, say $\mathcal{K}$, is called the a.s. limit (cluster) set of $\{\xi_T\}_{T>0}$. We write $\{\xi_T\}_{T>0}\to\to\mathcal{K}$.

If we deal with $C[0,1]$ endowed with the uniform topology (it is known to be separable), then $\{\xi_T\}_{T>0}\to\to\mathcal{K}$ iff

1. $\lim_{T\to\infty}\inf_{f\in\mathcal{K}}\|\xi_T-f\|=0$ a.s., and
2. for all $f\in\mathcal{K}$ $\ \liminf_{T\to\infty}\|\xi_T-f\|=0$ a.s.

Depending on the rate of growth of $\gamma(\cdot)$ the following variants of a.s. cluster sets for the family $\{\xi_T^\gamma\}_{T>0}$ exist:

(a) If $\gamma(T)=o(1)$, then the cluster set consists just from the zero function, which we denote by $\mathbf{0}$

$$\{\xi_T^\gamma\}\to\to\{\mathbf{0}\}.$$

(b) If $\gamma(T)\to c$, then the a.s. cluster set is a compact. Namely, the Strassen LIL holds

$$\{\xi_T^\gamma\}\to\to c^2\,\mathcal{S},$$

where $\mathcal{S}=\{f\in\mathbf{H},\int_0^1 f'^2\le 1\}$.

(c) If $\gamma(T) \to \infty$ in such a way that $\gamma(T) = o(\log\log T)$, then the following is true:

$$\{\xi_T^\gamma\} \to\to \mathcal{C}.$$

(d) If $\gamma(T) \to \infty$ in such a way that there is $c_0 > 0$ such that $c_0 \log\log T \leq \gamma(T)$ for large enough T, then the cluster set is empty. Namely, for any $f \in \mathcal{C}$ we have

$$\liminf_{T\to\infty} \|\xi_T^\gamma(\cdot) - f(\cdot)\| \geq \frac{c_0 \pi}{4} \quad a.s.$$

This scaling is too small to overpower natural fluctuations of the Wiener process, that is why the trajectories stop a.s. clustering around continuous functions.

2.2 Functional LIL for scaled stable Lévy processes.

In this section, we work in $D[0,1]$ endowed with the uniform topology, this is known non-separable topological space. The process X_α has no time-fixed jumps, therefore the uniform convergence is possible just to continuous functions. In this case, a.s. cluster sets $\mathcal{K}_h = \{f \in D[0,1] : \liminf_{T\to\infty} \|\frac{X_\alpha(T\cdot)}{T^{1/\alpha}h(T)} - f(\cdot)\| = 0\}$, if exist, are contained in $\mathcal{C}$.

Theorem 3 *Let* $h : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $h(0) = 0$ *and there exists* $c > 0$ *such that* $h(T) \leq c(\log\log T)^{-1/\alpha}$ *for* T *large enough. Then, for any* $f \in \mathcal{C}$ *the following holds*

$$\liminf_{T\to\infty} \left\| \frac{X_\alpha(T\cdot)}{T^{1/\alpha}h(T)} - f(\cdot) \right\| \geq \frac{K_\alpha^{1/\alpha}}{c} \quad a.s.,$$

where K_α *is as in (3).*

This statement corresponds to the case (d) for the Wiener process. This is a degenerate situation from the point of view of the functional LIL, fluctuations of the process overpower the scaling, it corresponds to the empty limit set.

If $h(\cdot)$ is such that $\lim_{T\to\infty} \frac{|X_\alpha(T)|}{T^{1/\alpha}h(T)} = 0$ in Fact 2, then the a.s. cluster set is not bigger than $\{\mathbf{0}\}$. It corresponds to the case (a) for the Wiener process (scaling is too strong).

We eliminate these two well-understood cases. Hence, our interest is focused on the set of scaling functions $h : \mathbf{R}^+ \to \mathbf{R}^+$ obeying conditions: $h(0) = 0$ and for any $c > 0$, any $\varphi : \int^\infty dx/\varphi(x) = \infty$ there exists $t_0 > 0$ such that for all $T \in (t_0, \infty)$

$$c(\log\log T)^{-1/\alpha} < h(T) \leq (\log T)^{1/\alpha}(\varphi(\log\log T))^{1/\alpha}.$$

Small deviations estimates from Theorem 1 give us the following statement.

Theorem 4 *For any* $f \in \mathcal{C}$*, any* $\delta \in (0,1)$ *we have*

$$\liminf_{T\to\infty} (\log\log T)^\delta \left\| \frac{X_\alpha(T\cdot)}{T^{1/\alpha}(\log\log T)^{\delta - 1/\alpha}} - f(\cdot) \right\| = K_\alpha^{1/\alpha} \quad a.s.$$

Comments:

1. From this statement it follows that if $\delta \in (0,1)$, then

$$\left\{ \frac{X_\alpha(T\cdot)}{T^{1/\alpha}(\log\log T)^{\delta-1/\alpha}} \right\} \to\to \mathcal{C}.$$

2. Take $\delta = 1/\alpha$, to obtain the following effect:

$$\left\{ \frac{X_\alpha(T\cdot)}{T^{1/\alpha}} \right\} \to\to \mathcal{C},$$

   despite the fact that $\frac{X_\alpha(T\cdot)}{T^{1/\alpha}} \stackrel{d}{=} X_\alpha(\cdot)$ for any $T > 0$. Under any fixed T we get càdlàg looking trajectories, nevertheless the gradual scaling (moving T to ∞) of the trajectories causes their clustering (almost all of them) around continuous functions. The same effect took place for W, see (c) case under $\gamma(T) = \sqrt{\log\log T}$.

3. We already mention that the uniform topology is not separable on $D[0,1]$, therefore the a.s. cluster set could be bigger if we consider the Skorokhod topology, which is separable on $D[0,1]$. But anyway the cluster set will contain $\mathcal{C}$ because the uniform convergence implies convergence in the Skorokhod topology on $D[0,1]$.

Small deviations estimates from Theorem 2 give us the following statement

Theorem 5 *For any f that belongs to*

$$\mathcal{C}^* = \left\{ f \in AC[0,1] : \ f(0) = 0, \ \|f'\| < \frac{2}{2-\alpha} \cdot (C(\alpha))^{-(\alpha-1)/\alpha} \right\}$$

we have

$$\liminf_{T\to\infty} (\log\log T) \left\| \frac{X_\alpha(T\cdot)}{(T/\log\log T)^{1/\alpha}\log\log T} - f(\cdot) \right\| = C' \quad a.s.,$$

where $C' \in [K_\alpha^{1/\alpha}, (C(\alpha))^{1/\alpha}]$, $C(\alpha)$ is from Theorem 2 and K_α is as in (3).

Comment:
From this statement it follows that the a.s. limit set of $\left\{ \frac{X_\alpha(T\cdot)}{T^{1/\alpha}(\log\log T)^{1-1/\alpha}} \right\}_{T>0}$ contains $\mathcal{C}^*$.

Proof. We modify the proofs of Theorem VIII.6 in [Ber96] and Theorem 17.1 in [Lif95]; for the lower bound we also use ideas of [Csá80].

Lower bound in Theorems 3, 4 and 5: Let $\delta \in [0,1]$, where $\delta = 0$ corresponds to Theorem 3, $\delta \in (0,1)$ to Theorem 4 and $\delta = 1$ to Theorem 5. Choose $T_k = \exp\{k(\log k)^{-3}\}$.

We start with the inequalities

$$\liminf_{T\to\infty} (\log\log T)^{\delta} \left\| \frac{X_\alpha(T\cdot)}{T^{1/\alpha}(\log\log T)^{\delta-\frac{1}{\alpha}}} - f(\cdot) \right\| \geq$$
$$\liminf_{k\to\infty} \frac{\inf_{[T_k,T_{k+1}]} \left\| X_\alpha(T\cdot) - f(\cdot)\, T^{1/\alpha}(\log\log T)^{\delta-\frac{1}{\alpha}} \right\|}{(T_{k+1}/\log\log T_{k+1})^{1/\alpha}} \geq$$
$$\liminf_{k\to\infty} \frac{\left\| X_\alpha(T_k\cdot) - f(\cdot) T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha} \right\|}{(T_{k+1}/\log\log T_{k+1})^{1/\alpha}}.$$

For the last inequality we used the following technical lemma

Lemma 2 *(i) For any increasing sequence $\{T_k\}_{k>0}$, any $f \in \mathcal{C}$ there exists $0 < M < \infty$ such that the following is true*

$$\inf_{[T_k,T_{k+1}]} \left\| X_\alpha(T\cdot) - f(\cdot) T^{1/\alpha}(\log\log T)^{\delta-1/\alpha} \right\| \geq \left\| X_\alpha(T_k\cdot) - f(\cdot) T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha} \right\| -$$
$$\|f\| \left(T_{k+1}^{1/\alpha}(\log\log T_{k+1})^{\delta-1/\alpha} - T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha} \right) -$$
$$M \cdot (1 - T_k/T_{k+1})^{1/2} (T_k)^{1/\alpha} \cdot (\log\log T_k)^{\delta-1/\alpha} \quad a.s.$$

(ii) For $T_k = \exp\{k(\log k)^{-3}\}$ the following holds

$$\lim_{k\to\infty} \frac{T_{k+1}^{1/\alpha}(\log\log T_{k+1})^{\delta-1/\alpha} - T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}}{(T_{k+1}/\log\log T_{k+1})^{1/\alpha}} = 0,$$

$$\lim_{k\to\infty} \frac{(1 - T_k/T_{k+1})^{1/2}(T_k)^{1/\alpha} \cdot (\log\log T_k)^{\delta-1/\alpha}}{(T_{k+1}/\log\log T_{k+1})^{1/\alpha}} = 0,$$

$$\lim_{k\to\infty} \frac{T_k}{T_{k+1}} = 1.$$

Proof. To proof (i), choose $\tau_k \in [T_k, T_{k+1}]$ such that

$$\left\| X_\alpha(\tau_k\cdot) - f(\cdot)\tau_k^{1/\alpha}(\log\log \tau_k)^{\delta-1/\alpha} \right\| = \inf_{[T_k,T_{k+1}]} \left\| X_\alpha(T\cdot) - f(\cdot) T^{1/\alpha}(\log\log T)^{\delta-1/\alpha} \right\|.$$

Then we need some cumbersome computations

$$\left\| X_\alpha(T_k\cdot) - f(\cdot) T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha} \right\| = \sup_{s\in[0,T_k]} |X_\alpha(s) - f(s/T_k) T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}| =$$

$$
\begin{aligned}
&\sup_{s\in[0,T_k/\tau_k]} |X_\alpha(s\tau_k) - f(s\tau_k/T_k)T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}| \le \\
&\sup_{s\in[0,T_k/\tau_k]} |X_\alpha(s\tau_k) - f(s)\tau_k^{1/\alpha}(\log\log \tau_k)^{\delta-1/\alpha}| + \\
&\sup_{s\in[0,T_k/\tau_k]} |f(s\tau_k/T_k)T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha} - f(s)\tau_k^{1/\alpha}(\log\log \tau_k)^{\delta-1/\alpha}| \le \\
&\|X_\alpha(\tau_k\cdot) - f(\cdot)\tau_k^{\frac{1}{\alpha}}(\log\log\tau_k)^{\delta-1/\alpha}\| + \sup_{s\in[0,\frac{T_k}{\tau_k}]}|f(s)| \left(\tau_k^{\frac{1}{\alpha}}(\log\log\tau_k)^{\delta-1/\alpha} - \right. \\
&\left. -T_k^{\frac{1}{\alpha}}(\log\log T_k)^{\delta-1/\alpha}\right) + \sup_{s\in[0,T_k/\tau_k]}|f(s\tau_k/T_k) - f(s)|\, T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}.
\end{aligned}
$$

Let us show that there exists $0 < M < \infty$ such that $\|f(T_k/\tau_k\cdot) - f(\cdot)\| \le M \cdot (1 - T_k/T_{k+1})^{1/2}$. Note that $(1 - T_k/T_{k+1})^{1/2} < 1$.

It is known that $\mathbf{H}$ is dense in $\mathcal{C}$. Thus, for any $k > 0$ there exists $f_k \in \mathbf{H}$ such that $\|f_k(\cdot) - f(\cdot)\| \le (1 - T_k/T_{k+1})^{1/2}$. We can write

$$
\begin{aligned}
&\|f(T_k/\tau_k\cdot) - f(\cdot)\| \le 2\|f_k(\cdot) - f(\cdot)\| + \|f_k(T_k/\tau_k\cdot) - f_k(\cdot)\| \le \\
&(2 + \|f_k'\|_{L_2}) \cdot (1 - T_k/T_{k+1})^{1/2}.
\end{aligned}
$$

For the last step we used: for any $0 \le a \le 1$, any $s \in [0,1]$ we have $|f_k(a\,s) - f_k(s)| \le \|f_k'\|_{L_2}(1-a)^{1/2}$, that is easy to prove by Schwarz's inequality.

The rest is obvious.

To prove (ii), note

$$
1 \ge \frac{T_k}{T_{k+1}} = \exp\left\{\frac{k}{(\log k)^3} - \frac{k+1}{(\log(k+1))^3}\right\} \ge \exp\left\{-\frac{1}{(\log k)^3}\right\},
$$

and $\log\log T_k = \log k\,(1 + o(1))$. It is left just to make computations.

■

Let us show that

$$
\liminf_{k\to\infty} \frac{\left\|X_\alpha(T_k\cdot) - f(\cdot)T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}\right\|}{(T_{k+1}/\log\log T_{k+1})^{1/\alpha}} \ge K_\alpha^{1/\alpha}.
$$

Take $A > 0$. We use the Anderson inequality, self-similatity, and estimate (3) to obtain

$$
\begin{aligned}
&\mathbf{P}\left\{\left\|X_\alpha(T_k\cdot) - f(\cdot)T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}\right\| < A(T_{k+1}/\log\log T_{k+1})^{1/\alpha}\right\} \le \\
&\mathbf{P}\left\{\|X_\alpha(T_k\cdot)\| < A\,(T_{k+1}/\log\log T_{k+1})^{1/\alpha}\right\} \le
\end{aligned}
$$

$$\mathbf{P}\left\{\|X_\alpha(\cdot)\| < A\left(\frac{T_{k+1}}{T_k}\right)^{1/\alpha}(\log\log T_{k+1})^{-1/\alpha}\right\} \le$$
$$\le \exp\left\{-\frac{K_\alpha}{A^\alpha}\frac{T_k}{T_{k+1}}\log\log T_{k+1}(1+o(1))\right\}.$$

Using the particular form of $\{T_k\}_{k>0}$, we get

$$\mathbf{P}\left\{\left\|X_\alpha(T_k\cdot) - f(\cdot)T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}\right\| < A\,(T_{k+1}/\log\log T_{k+1})^{1/\alpha}\right\} \le$$
$$\exp\left\{-\frac{K_\alpha}{A^\alpha}\log(k/(\log k)^3)(1+o(1))\right\}.$$

Choose $A = (K_\alpha/(1+\epsilon))^{1/\alpha}$, $\epsilon > 0$ obtain

$$\mathbf{P}\left\{\left\|X_\alpha(T_k\cdot) - f(\cdot)T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}\right\| < \left(\frac{K_\alpha}{1+\epsilon}\cdot\frac{T_{k+1}}{\log\log T_{k+1}}\right)^{1/\alpha}\right\} \le \left(\frac{(\log k)^3}{k}\right)^{(1+\epsilon)(1+o(1))}.$$

Use the Borel-Cantelli lemma and obtain that for any $f \in \mathcal{C}$, any $\epsilon > 0$ the following holds

$$\liminf_{k\to\infty}\frac{\left\|X_\alpha(T_k\cdot) - f(\cdot)T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}\right\|}{(T_{k+1}/\log\log T_{k+1})^{1/\alpha}} \ge \left(\frac{K_\alpha}{1+\epsilon}\right)^{1/\alpha} \quad a.s.$$

To conclude the proof, tend $\epsilon \to 0$.

Addition to Theorem 3: The scheme of the proof is the same for $h(T) = o((\log\log T)^{-1/\alpha})$ as $T \to \infty$. The difference is just in the first inequality

$$\liminf_{T\to\infty}\left\|\frac{X_\alpha(T\cdot)}{T^{1/\alpha}h(T)} - f(\cdot)\right\| \ge \liminf_{k\to\infty}\frac{\inf_{[T_k,T_{k+1}]}\left\|X_\alpha(T\cdot) - f(\cdot)\,T^{1/\alpha}h(T)\right\|}{(T_{k+1}/\log\log T_{k+1})^{1/\alpha}}.$$

Lemma 2 could be modified correspondingly.

Upper bound for Theorem 4 and Theorem 5: Choose $T_k = \exp\{k^\gamma\}$, $\gamma > 1$. Consider events

$$D_k(A) = \left\{\left\|\frac{X_\alpha(T_k\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}} - f(\cdot)\right\| \le A\,\frac{1}{(\log\log T_k)^\delta}\right\},$$

where $0 < A < \infty$, $\delta \in (0,1]$. Let us estimate

$$\mathbf{P}\{D_k(A)\} = \mathbf{P}\left\{\left\|X_\alpha(\cdot) - f(\cdot)(\log\log T_k)^{\delta-1/\alpha}\right\| \le A\,\frac{1}{(\log\log T_k)^{1/\alpha}}\right\}.$$

For $\delta \in (0,1)$, we use the lower bound of Theorem 1

$$\mathbf{P}\left\{\left\|X_\alpha(\cdot) - f(\cdot)(\log\log T_k)^{\delta-1/\alpha}\right\| \le A\,\frac{1}{(\log\log T_k)^{1/\alpha}}\right\} \ge$$
$$\exp\left\{-\frac{K_\alpha}{A^\alpha}\log\log T_k\,(1+o(1))\right\} = \exp\left\{-\frac{K_\alpha}{A^\alpha}\gamma\log k\,(1+o(1))\right\}.$$

Putting $A_\gamma = (K_\alpha\gamma)^{1/\alpha}$, we obtain $\mathbf{P}\{D_k(A_\gamma)\} \ge 1/k$. Thus,

$$\sum_{k=1}^{\infty}\mathbf{P}\ \{D_k(A_\gamma)\} = \infty. \tag{9}$$

For $\delta = 1$, we use the lower bound of Theorem 2 that holds for any $\|f'\| < \frac{2}{2-\alpha}\cdot\frac{1}{A^{\alpha-1}}$ and obtain

$$\mathbf{P}\ \{D_k(A)\} = \mathbf{P}\left\{\left\|X_\alpha(\cdot) - f(\cdot)(\log\log T_k)^{1-1/\alpha}\right\| \le A\,\frac{1}{(\log\log T_k)^{1/\alpha}}\right\} \ge$$
$$\exp\left\{-\frac{C(\alpha)}{A^\alpha}\log\log T_k\,(1+o(1))\right\} = \exp\left\{-\frac{C(\alpha)}{A^\alpha}\gamma\log k\,(1+o(1))\right\}$$

Put $A_\gamma = (C(\alpha)\gamma)^{1/\alpha}$, and obtain (9).

We could not use the Borel-Cantelli lemma directly because the events $\{D_k\}$ are dependent. To overcome this difficulty we decompose the process into a sum of independent processes:

$$X_\alpha(T_k\cdot) = Y_k(\cdot) + Z_k(\cdot)\ \ a.s., \tag{10}$$

where

$$Y_k(s) = \begin{cases} X_\alpha(T_k s), & s \in [0, \frac{T_{k-1}}{T_k}] \\ X_\alpha(T_{k-1}), & s \in [\frac{T_{k-1}}{T_k}, 1] \end{cases}$$

and

$$Z_k(s) = \begin{cases} 0, & s \in [0, \frac{T_{k-1}}{T_k}] \\ X_\alpha(T_k s) - X_\alpha(T_{k-1}), & s \in [\frac{T_{k-1}}{T_k}, 1]. \end{cases}$$

It is easy to see that $Z_1(\cdot), Z_2(\cdot), ...Z_k(\cdot), Z_{k+1}(\cdot), ...$ are independent processes (they are constructed by using increments of X_α at non-intersecting intervals).

Let us prove the following

$$\sum_{k=1}^{\infty}\mathbf{P}\left\{\left\|\frac{Z_k(\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}} - f(\cdot)\right\| \le (1+\epsilon)A_\gamma\frac{1}{\log\log T_k}\right\} = \infty. \tag{11}$$

We use

$$\mathbf{P}\left\{\left\|\frac{Z_k(\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}}-f(\cdot)\right\|\le(1+\epsilon)A_\gamma\frac{1}{(\log\log T_k)^\delta}\right\}\ge$$
$$\mathbf{P}\left\{\left\|\frac{X_\alpha(T_k\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}}-f(\cdot)\right\|+\frac{\|Y_k(\cdot)\|}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}}\le(1+\epsilon)A_\gamma\frac{1}{(\log\log T_k)^\delta}\right\}\ge$$
$$\mathbf{P}\left[\left\{\left\|\frac{X_\alpha(T_k\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}}-f(\cdot)\right\|\le\frac{A_\gamma}{(\log\log T_k)^\delta}\right\}\cap\left\{\frac{\|Y_k(\cdot)\|}{T_k^{1/\alpha}(\log\log T_k)^{-\frac{1}{\alpha}}}\le\epsilon\,A_\gamma\right\}\right]\ge$$
$$\mathbf{P}\left\{D_k(A_\gamma)\right\}-\mathbf{P}\left\{\frac{\|Y_k(\cdot)\|}{T_k^{1/\alpha}(\log\log T_k)^{-1/\alpha}}\ge\epsilon A_\gamma\right\}.$$

It is left to prove that the second term could be majorized by a term of convergent series. Take arbitrary $\epsilon>0$. Consider the events

$$C_k(\epsilon A_\gamma)=\left\{\frac{\|Y_k(\cdot)\|}{T_k^{1/\alpha}(\log\log T_k)^{-1/\alpha}}>\epsilon\,A_\gamma\right\}.$$

Now we need a large deviation result (cf. p.238, [Ber96])

$$\mathbf{P}\left\{\|X_\alpha(\cdot)\|>x\right\}=Kx^{-\alpha}(1+o(1))\ \ as\ x\to\infty,$$

what is true for some $0<K<\infty$. Using $\|Y_k(\cdot)\|=\|X_\alpha(T_{k-1}\cdot)\|$ a.s. and the self-similarity we write

$$\mathbf{P}\left\{C_k(\epsilon A_\gamma)\right\}=\mathbf{P}\left\{\frac{\|X_\alpha(T_{k-1}\cdot)\|}{T_k^{1/\alpha}(\log\log T_k)^{-1/\alpha}}>\epsilon\,A_\gamma\right\}=$$
$$\mathbf{P}\left\{\frac{\|X_\alpha(T_{k-1}\cdot)\|}{T_{k-1}^{1/\alpha}}>\epsilon\,A_\gamma\left(\frac{T_k}{T_{k-1}}\right)^{1/\alpha}(\log\log T_k)^{-1/\alpha}\right\}=\epsilon\,A_\gamma\,K\cdot\frac{T_{k-1}}{T_k}\log\log T_k\,(1+o(1)).$$

Now we use

$$\sum_{k=1}^\infty\frac{T_{k-1}}{T_k}\log\log T_k=\gamma\sum_{k=1}^\infty\frac{\log k}{\exp\{\gamma k^{\gamma-1}\}}(1+o(1))<\infty.$$

Thus, for any $\epsilon>0$, any $\gamma>1$ we have

$$\sum_{k=1}^\infty\mathbf{P}\left\{C_k(\epsilon A_\gamma)\right\}<\infty.$$

Using Borel-Cantelli lemma we also obtain

$$\limsup_{k\to\infty} (\log\log T_k)^{1/\alpha} \frac{\|Y_k(\cdot)\|}{T_k^{1/\alpha}} = 0.$$

So, this and (9) prove (11) and the events there are independent. We apply the Borel-Cantelli lemma and obtain

$$\liminf_{k\to\infty} (\log\log T_k)^{\delta} \left\| \frac{Z_k(\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}} - f(\cdot) \right\| \le (1+\epsilon)A_\gamma.$$

It is left just to use the elementary relations

$$\begin{aligned}
&\liminf_{T\to\infty} (\log\log T)^{\delta} \left\| \frac{X_\alpha(T\cdot)}{T^{1/\alpha}(\log\log T)^{\delta-1/\alpha}} - f(\cdot) \right\| \le \\
&\liminf_{k\to\infty} (\log\log T_k)^{\delta} \left\| \frac{X_\alpha(T_k\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}} - f(\cdot) \right\| \le \\
&\liminf_{k\to\infty} (\log\log T_k)^{\delta} \left[\left\| \frac{Z_k(\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}} - f(\cdot) \right\| + \frac{\|Y_k(\cdot)\|}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}} \right] \le \\
&\liminf_{k\to\infty} (\log\log T_k)^{\delta} \left\| \frac{Z_k(\cdot)}{T_k^{1/\alpha}(\log\log T_k)^{\delta-1/\alpha}} - f(\cdot) \right\| + \limsup_{k\to\infty} (\log\log T_k)^{1/\alpha} \frac{\|Y_k(\cdot)\|}{T_k^{1/\alpha}}.
\end{aligned}$$

Tending $\gamma \to 1$ and $\epsilon \to 0$, we obtain the upper bound.

∎

Open questions:

1. Wide field of action is to find a.s. limit sets in the case of scaling functions

   $$(\log\log T)^{1-1/\alpha} < h(T) \le (\log T)^{1/\alpha}(\varphi(\log\log T))^{1/\alpha},$$

   where φ is as in (7). From the proof we see that a positive result (the a.s. limit set is wider than $\{\mathbf{0}\}$) requires a good lower bound of $\mathbf{P}\{\|X_\alpha - \lambda f\| < r\}$ under $\lambda r^{\alpha-1} \to \infty$, and a negative result (the a.s. limit set is $\{\mathbf{0}\}$) would require a good upper bound of the same probability.

2. It is interesting to study the functional LIL in the Skorokhod topology. We already mention that the a.s. cluster set will contain the a.s. cluster set under the uniform convergence. It is also possible that the set of admissible scaling functions is wider.

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